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A survey on Nahm transform

Marcos Jardim*

Department of Mathematics and Statistics, University of Massachusetts at Amherst, Amherst, MA 01003-9305, USA

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Abstract

We review the construction known as the Nahm transform in a generalised context, which includes all the examples of this construction already described in the literature. The Nahm transform for translation invariant instantons on $\mathbb{R}^4$ is presented in an uniform manner. We also briefly analyse two new examples, the second of which being the first example involving a four-manifold that is not hyperkähler.

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1. Introduction

Since the appearance of the Yang–Mills equation on the mathematical scene in the late 1970s, its anti-self-dual (ASD) solutions have been intensively studied. The first major result in the field was the ADHM construction of instantons on $\mathbb{R}^4$ [2]. Soon after that, Nahm adapted the ADHM construction to obtain the *time-invariant* ASD solutions of the Yang–Mills equations, the so-called monopoles [24,25]. Nahm found a correspondence between solutions of the anti-self-duality equations which are invariant under translations in one direction and solutions of the anti-self-duality equations which are invariant under translations in three directions. His physical arguments were formalised in a beautiful paper by Hitchin [18].

* Tel.: +1-413-5453128; fax: +1-413-5451801.
E-mail address: jardim@math.umass.edu (M. Jardim).

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It was later realised that these constructions are two examples of a much more general framework. This was first pointed out by Corrigan and Goddard in [10], and further elaborated in papers by Braam and van Baal [8] (who coined the term "Nahm transform") and by Nakajima [26].

The Nahm transform was initially conceived as a correspondence between solutions of the anti-self-duality equations which are invariant under dual subgroups of translations of $\mathbb{R}^4$, and many such correspondences have been described in the literature (see Section 3). The first goal of this paper, pursued in Section 2, is to show that the transform can be set-up in a much larger class of four-manifolds, namely spin manifolds of non-negative scalar curvature. It can be characterised as nonlinear version of the Fourier transform, which takes vector bundles provided with anti-self-dual connections over a four-dimensional manifold into vector bundles with connections over a *dual* manifold. If further geometric structures are available one can easily show that the transformed connection satisfies certain natural differential conditions. In particular, if the original manifold admits a hyperkähler metric, then the transformed connection is a quaternionic instanton.

We then list all instances of the Nahm transform described in the literature, adding two new examples. The second one, concerning instantons on the four-sphere, is of particular interest, for it involves a four-dimensional manifold which does not admit a complex structure.

This paper is written with a wider audience of in mind, so arguments familiar to experts are presented in detail. We focus on the mathematical aspects and precise mathematical statements surrounding the Nahm transform. There is an extensive physical literature relating Nahm transform and fundamental problems in physics, like quark confinement in QCD and string dualities. For the reader interested in these issues, we recommend for instance [16] (among other papers by van Baal) for the relevance of Nahm transform in QCD on the lattice and [9,11,23,32] for the relations between Nahm transform and string theory. Another interesting related topic that is out of this survey is the role of Nahm transform in noncomutative gauge theories [1,27].

2. The Nahm transform

Let (M, g_M) be a smooth oriented Riemannian spin four-manifold with non-negative scalar curvature ($R_M \geq 0$). For simplicity, we assume that M is compact. We denote by $S^{\pm}$ the spinor bundles of positive and negative chirality.

Consider a Hermitian vector bundle E over M, and let A be an unitary anti-self-dual connection on E; more precisely, its curvature F_A satisfies the following condition:

$$*F_A = -F_A, \tag{1}$$

where $*$ denotes the Hodge star operator.

Now let T be a smooth manifold parametrising a family of (gauge equivalence classes of) anti-self-dual connections on a fixed complex vector bundle $F \to M$. In other words, each $t \in T$ corresponds to an anti-self-dual connection B_t on the bundle F. Typically, we can think of T as a (submanifold of a) moduli space of irreducible anti-self-dual connections on $F \to M$. Note also that the Riemannian metric on M induces a natural L^2-metric g_T on T.

The *Nahm transform from M to T* is a mechanism that transforms Hermitian vector bundles with unitary anti-self-dual connections on M into Hermitian vector bundles with unitary connections on T. If T parameterises a family of flat connections over M, we will say that the transform is *flat*; otherwise, we will say that the Nahm transform is *non-flat*.

Let us now describe the transform in detail. On the tensor bundle $E \otimes F$, we have a twisted family of anti-self-dual connections $A_t = A \otimes \mathbf{1}_F + \mathbf{1}_E \otimes B_t$. We further assume that this family is *1-irreducible*, in the sense that:

$$\nabla_{A_t} s = 0 \Rightarrow s = 0 \quad \forall t \in T,$$

i.e. the tensor bundle $E \otimes F$ has no covariantly constant sections.

We consider the family of coupled Dirac operators:

$$D_{A_t} : L^2_p(E \otimes F \otimes S^+) \to L^2_{p-1}(E \otimes F \otimes S^-)$$

let $D^*_{A_t}$ denote the dual Dirac operator. The Dirac Laplacian $D^*_{A_t} D_{A_t}$ is related to the trace Laplacian $\nabla^*_{A_t} \nabla_{A_t}$ via the Weitzenböck formula:

$$D^*_{A_t} D_{A_t} = \nabla^*_{A_t} \nabla_{A_t} - F^+_{A_t} + \tfrac{1}{4} R_M. \tag{2}$$

Applying (2) to a section $s \in L^2_p(E \otimes F \otimes S^+)$, and integrating by parts, we obtain

$$\|D_{A_t} s\|^2 = \|\nabla_{A_t} s\|^2 + \frac{1}{4} \int_M R_M \langle s, s \rangle \geq 0 \tag{3}$$

with equality if and only if $s = 0$, since $F^+_{A_t} = 0$ and $R_M \geq 0$. Therefore, we conclude that $\ker D_{A_t} = \{0\}$ for all $t \in T$.

This means that $\hat{E} = -\mathrm{Index}\{D_{A_t}\}$ is a well-defined Hermitian vector bundle over T; the fibre $\hat{E}_t$ is given by coker D_{A_t}.

Furthermore, letting $\hat{H}$ denote the trivial Hilbert bundle over T with fibres given by $L^2_{p-1}(E \otimes F \otimes S^-)$, one can also define an unitary connection $\hat{A}$ via the *projection formula*:

$$\nabla_{\hat{A}} = P \underline{d} \iota, \tag{4}$$

where $\iota : \hat{E} \to \hat{H}$ denotes the natural inclusion, $\underline{d}$ denotes the trivial covariant derivative on $\hat{H}$ and $P : \hat{H} \to \hat{E}$ denotes the orthogonal projection induced by the L^2 inner product; at each $t \in T$, this projection can be expressed in the following way:

$$P(t) = \mathbf{1}_{\hat{H}} - D_{A_t} G_{A_t} D^*_{A_t}, \tag{5}$$

where $G_{A_t} = (D^*_{A_t} D_{A_t})^{-1}$ is the Green's operator for the Dirac Laplacian.

Notice that if $t, t' \in T$ are such that the corresponding connections B_t and $B_{t'}$ are gauge equivalent, then clearly A_t and $A_{t'}$ are also gauge equivalent. Hence there is a natural isomorphism $\ker D^*_{A_t} \xrightarrow{\sim} \ker D^*_{A_{t'}}$, and the index bundle $\hat{E}$ descends to a bundle on the quotient $T/\mathcal{G}$, where $\mathcal{G}$ denotes the group of gauge transformations of F. For this reason, we assume from now on that T parameterises a family of gauge equivalence classes of irreducible anti-self-dual connections on F.

The pair $(\hat{E}, \hat{A})$ is called the *Nahm transform* of (E, A).

Remark 1. The key necessary and sufficient condition for the transform to work is the vanishing of the kernel of the Dirac operators D_{A_t} for all $t \in T$. This means that the non-negativity condition on the scalar curvature R_M can be weakened. Indeed, consider the following bilinear Hermitian pairing on $L^2(E \otimes F \otimes S^+)$:

$$\{s_1, s_2\} := \int_M R_M \langle s_1, s_2 \rangle, \quad s_1, s_2 \in L^2(E \otimes F \otimes S^+).$$

Using theWeitzenböck formula (2), it is easy to see that $\ker D_{A_t} = 0$ if and only if $\{s, s\} \geq -4\|\nabla_{A_t} s\|^2$ for all $s \in L^2(E \otimes F \otimes S^+)$ and all $t \in T$, with equality if and only if $s = 0$.

Lemma 2. *If A and A' are two gauge equivalent connections on a vector bundle $E \to X$, then $\hat{A}$ and $\hat{A}'$ are gauge equivalent connections on the transformed bundle $\hat{E} \to Y$.*

In other words, the Nahm transform yields a well-defined map from the moduli space of gauge equivalence classes of anti-self-dual connections on $E \to M$ into the space of gauge equivalence classes of connections on $\hat{E} \to T$.

Proof. Since A and A' are gauge equivalent, there is a bundle automorphism $h : E \to E$ such that $\nabla'_A = h^{-1} \nabla_A h$. Take $g = h \otimes \mathbf{1}_F \in \mathrm{Aut}(E \otimes F)$, so that $\nabla_{A'_t} = g^{-1} \nabla_{A_t} g$, hence $D^*_{A'_t} = g^{-1} D^*_{A_t} g$, for all $t \in T$. Thus if $\{\Psi_i\}$ is a basis for $\ker D^*_{A_t}$, then $\{\Psi'_i = g^{-1}\Psi_i\}$ is a basis for $\ker D^*_{A'_t}$. So g can also be regarded as an automorphism of the transformed bundle $\hat{E}$. It is then easy to see that:

$$\nabla_{\hat{A}'} = P' \underline{d} \iota' = (g^{-1} P' g) \underline{d} (g^{-1} \iota' g) = g^{-1} \nabla_{\hat{A}} g$$

since $\underline{d} g^{-1} = 0$, for $g = h \otimes \mathbf{1}_E$ does not depend on t. □

The Nahm transformed connection $\hat{A}$ was defined above in a rather coordinate-free manner. For many calculations, it is important to have a more explicit description. First note that the rank of the transformed bundle $\hat{E}$ is just the index of the Dirac operator $D^*_{A_t}$ for some $t \in T$, so it is given by

$$\hat{r} = \mathrm{rank}\, \hat{E} = -\int_M \mathrm{ch}(E) \cdot \mathrm{ch}(F) \cdot \left(1 - \frac{1}{24} p_1(M)\right), \tag{6}$$

where $p_1(M)$ denotes the first Pontryagin class of M. Recall that since M is a spin four-manifold, then

$$p = \frac{1}{24} p_1(M)[M] = \frac{1}{192\pi^2} \int_M \mathrm{Tr}(R_M \wedge R_M)$$

is an even integer (so-called $\hat{\mathfrak{a}}$-genus of M).

Now let $\{\Psi_i = \Psi_i(x; t)\}_{i=1}^{\hat{r}}$ be linearly independent solutions of the Dirac equation $D^*_{A_\xi} \Psi_i = 0$. We can assume that $\langle \Psi_i, \Psi_j \rangle = \delta_{ij}$, where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product on $\hat{H}$. Clearly, $\{\Psi_i\}_{i=1}^{\hat{r}}$ forms a local orthonormal frame for $\hat{E}$. In this choice of trivialisation, the components of the connection matrix $\hat{A}$ can be written in the following way:

$$\hat{A}_{ij} = \langle \Psi_i, \nabla_{\hat{A}} \Psi_j \rangle = \langle \Psi_i, \underline{d} \Psi_j \rangle = \int_M \Psi_i(x; t)^\dagger \bullet \frac{\mathrm{d}}{\mathrm{d}t} \Psi_j(x; t)\, \mathrm{d}^4 x, \tag{7}$$

where $\bullet$ denotes Clifford multiplication.

In this trivialisation, the curvature can be expressed as follows:

$$(F_{\hat{A}})_{ij} = \langle \Psi_i, \nabla_{\hat{A}} \nabla_{\hat{A}} \Psi_j \rangle = \langle \Psi_i, \underline{d} P \underline{d} \Psi_j \rangle = \langle \Psi_i, \underline{d} D_{A_t} G_{A_t} D^*_{A_t} \underline{d} \Psi_j \rangle$$
$$= -\langle D^*_{A_t} \underline{d} \Psi_i, G_{A_t} D^*_{A_t} \underline{d} \Psi_j \rangle.$$

We define $\Delta = [D^*_{A_t}, \underline{d}]$; this is an algebraic operator acting as

$$\Delta : L^2(M \times T, \pi_1^*(E \otimes F \otimes S^-)) \to L^2(M \times T, \pi_1^*(E \otimes F \otimes S^+) \otimes \pi_2^* \Omega^1_T),$$

where π_1 and π_2 are the projections of $M \times T$ onto the first and second factors, respectively. More precisely, this operator can be expressed in terms of Clifford multiplication; in local coordinates:

$$\Delta = \sum_{k=1}^{\dim T} \delta_k(x; t)\, \mathrm{d}t_k,$$

where $\delta_k(x; t)$ is a local section of $\pi_1^*(E \otimes F \otimes S^-)$. With this in mind, we conclude that:

$$\Delta(\Psi) = \sum_{k=1}^{\dim T} \delta_k(x; t) \bullet \Psi\, \mathrm{d}t_k = \Delta \bullet \Psi,$$

where $\bullet$ denotes Clifford multiplication. Clearly, if $\Psi \in \ker D^*_{A_t}$, then $D^*_{A_t} \underline{d} \Psi = \Delta \bullet \Psi$, Therefore, we have

$$(F_{\hat{A}})_{ij} = -\langle \Delta \bullet \Psi_i, G_{A_t}(\Delta \bullet \Psi_j) \rangle. \tag{8}$$

It is important to note that the transformed connection $\hat{A}$ is smooth, but since the parameter space T might not be compact, $\hat{A}$ might not have finite L^2-norm (i.e. finite Yang–Mills action).

2.1. *The topology of the transformed bundle*

Let us now study the topological invariants of the transformed bundle. Recall that one can define a *universal bundle with connection* over the product $M \times T$ in the following way [3]. Let $\mathcal{A}$ denote the set of all connections on F, and let $\mathcal{G}$ denote the group of gauge transformations (i.e. bundle automorphims). Moreover, let G denote the structure group of F, so that F can be associated with a principal G_E bundle P over M by means of some representation $\rho : G \to \mathbb{C}^n$, where $n = \operatorname{rank} F$. $\mathcal{G}$ acts on $F \times \mathcal{A}$ by $g(p, A) = (g(p), g(A))$; This action has no fixed points, and it yields a principal $\mathcal{G}$-bundle $E \times \mathcal{A} \to \mathcal{Q}$, where $\mathcal{Q} = E \times \mathcal{A}/\mathcal{G}$.

The structure group G also acts on $E \times \mathcal{A}$, and since this action commutes with the one by $\mathcal{G}$, G acts on $\mathcal{Q}$. Moreover, the G-action on $\mathcal{Q}^{\mathrm{ir}} = E \times \mathcal{A}^{\mathrm{ir}}/\mathcal{G}$ has no fixed points, where $\mathcal{A}^{\mathrm{ir}}$ denotes the set of irreducible connections on F. We end up with a principal G bundle $\mathcal{Q}^{\mathrm{ir}} \to M \times (\mathcal{A}^{\mathrm{ir}}/\mathcal{G})$, and we denote by $\tilde{\mathbb{P}}$ the associated vector bundle $\mathcal{Q}^{\mathrm{ir}} \times_\rho \mathbb{C}^n$. Since T is a submanifold of $\mathcal{A}^{\mathrm{ir}}/\mathcal{G}$, we define the *Poincaré bundle* $\mathbb{P} \to M \times T$ as the restriction of $\tilde{\mathbb{P}}$.

The principal G bundle $\mathcal{Q}^{\text{ir}}$ also has a natural connection $\tilde{\omega}$, constructed as follows. The space $E\times\mathcal{A}^{\text{ir}}$ has a Riemannian metric which is equivariant under $G\times\mathcal{G}$, so that it descends to a G-equivariant metric on $\mathcal{Q}^{\text{ir}}$. The orthogonal complements to the orbits of G yields the connection $\tilde{\omega}$. Passing to the associated vector bundle $\tilde{\mathbb{P}}$ and restricting it to $M \times T$ gives a connection ω on the *Poincaré bundle* $\mathbb{P}$. The pair $(\mathbb{P}, \omega)$ is universal in the sense that $(\mathbb{P}, \omega)|_{M\times\{t\}} \simeq (F, B_t)$ [3].

The Atiyah–Singer index theorem for families allows us to compute the Chern character of the transformed bundle via the formula:

$$\operatorname{ch}\hat{E} = -\int_M \operatorname{ch}(E)\cdot\operatorname{ch}(\mathbb{P})\cdot\left(1-\frac{1}{24}p_1(M)\right), \tag{9}$$

where the minus sign is needed because $\hat{E}$ is the bundle of cokernels. The curvature Ω of the Poincaré connection ω can be easily computed, see [3]. In examples, that can then be used to compute the Chern character of $\mathbb{P}$.

2.2. *Differential properties of transformed connection*

Since the expression (8) for the curvature of the transformed connection does not depend explicitly on the curvature of the original connection A, it is in general very hard to characterise any particular properties of $F_{\hat{A}}$.

For instance, when the parameter space T is four-dimensional, one would like to know whether $F_{\hat{A}}$ is anti-self-dual. This seems to be a very hard question in general; we now offer a few positive results.

First, note that the algebraic operator $\Delta = [D_{A_t}, \underline{d}]$ can also be thought as a section of the bundle $\pi_1^*\mathcal{L}\otimes\pi_2^*\Omega^1_T$, where $\mathcal{L} = \operatorname{End}(E\otimes F\otimes S^-)$.

Proposition 3. *If* $[G_{A_t}, \Delta] = 0$, *then* $F_{\hat{A}}$ *is proportional to* $\Delta\wedge\Delta$ *as a two-form over the parameter space T. In particular, if T is four-dimensional,* $F_{\hat{A}}$ *is anti-self-dual if and only if* $\Delta\wedge\Delta$ *is a section of* $\pi_1^*\mathcal{L}\otimes\pi_2^*\Omega^{2,-}_T$.

Proof. If $G_{A_t}\Delta = \Delta G_{A_t}$, it follows from (8) that:

$$(F_{\hat{A}})_{ij} = -\langle\Delta\bullet\Psi_i, \Delta\bullet(G_{A_t}\Psi_j)\rangle = -\langle\Delta\bullet\Delta\bullet\Psi_i, G_{A_t}\Psi_j\rangle.$$

It is then easy to see from the last expression that each component $(F_{\hat{A}})_{ij}$ is proportional to $\Delta\wedge\Delta$ as a two-form over T. □

When M is a Kähler or hyperkähler manifold, complex analytic methods can also be useful. We turn to two well-known results concerning these cases.

Proposition 4. *If M and T are Kähler manifolds, then the transformed bundle* $\hat{E}$ *has a natural complex structure, which is compatible with* $\hat{A}$. *In particular, the curvature of the transformed connection is of type* $(1, 1)$.

It is important to recall that if M is a Kähler manifold, then all connected components of the moduli space of anti-self-dual connections on M are also Kähler. We include an outline of the proof of this well-known result for the sake of completeness, and for the convenience of the reader.

Proof. The anti-self-dual connection A_t induces a holomorphic structure on the tensor bundle $E \otimes F$, and the Dirac operators can be written in terms of the Dolbeault operators in the following manner:

$$D_{A_t} = 2(\bar{\partial}_{A_t} - \bar{\partial}^*_{A_t}) \quad \text{and} \quad D^*_{A_t} = 2(\bar{\partial}^*_{A_t} - \bar{\partial}_{A_t}).$$

Therefore Hodge theory gives identifications for each $t \in T$:

$$\ker D_{A_t} = \ker \bar{\partial}_{A_t} \oplus \ker \bar{\partial}^*_{A_t} = H^0(M, E \otimes F) \oplus H^2(M, E \otimes F),$$
$$\ker D^*_{A_t} = \ker \bar{\partial}^*_{A_t} \cap \ker \bar{\partial}_{A_t} = H^1(M, E \otimes F).$$

This means that $\hat{E}$ can be identified (as a smooth vector bundle) with the cohomology of the family Dolbeault complex:

$$E \otimes F \stackrel{\bar{\partial}_{A_t}}{\to} E \otimes F \otimes \Omega^{0,1}_M \stackrel{\bar{\partial}_{A_t}}{\to} E \otimes F \otimes \Omega^{0,2}_M .$$

General theory [13, pp. 79–80] then implies that $\hat{E}$ also has a holomorphic structure, with which the connection $\hat{A}$ defined via the projection formula (4) is compatible. □

Recall that a Riemannian four-manifold M is said to be *hyperkähler* if its holonomy group is contained in Sp(1). This implies that M carries three almost complex structures (I, J, K) which are parallel with respect to the Levi-Civita connection and satisfy quaternionic relations $IJ = -JI = K$.

A *quaternionic instanton* is a connection A on a complex vector bundle V over a hyperkähler manifold T whose curvature F_A is of type $(1, 1)$ with respect to all complex structures [5]. In particular, if T is four-dimensional then a quaternionic instanton is just an anti-self-dual connection.

Proposition 5. *If M and T are hyperkähler manifolds, then the transformed connection is a quaternionic instanton. In particular, if T is four-dimensional then $\hat{A}$ is anti-self-dual.*

As in Proposition 4, the hypothesis here are slightly redundant, for if M is hyperkähler, then all connected components of the moduli space of anti-self-dual connections on M are also hyperkähler.

Proof. Each choice of a Kähler structure on M induces a choice of a Kähler structure on T; by Proposition 4, $F_{\hat{A}}$ is of type $(1, 1)$ with respect to this structure. Thus $F_{\hat{A}}$ is of type $(1, 1)$ with respect to all Kähler structures on T, which means that $\hat{A}$ is a quaternionic instanton. □

Since the only compact four-dimensional hyperkähler manifolds are the four-torus and the K3-surface, this last result seems to have a rather limited applicability. However, as we will argue in Section 3, Proposition 5 can also be used to define a Nahm transform for instantons over hyperkähler ALE spaces.

It is also important to mention that a higher-dimensional generalisation of the Nahm transform for quaternionic instantons over hyperkähler manifolds has been described by Bartocci et al. [5].

Remark 6. Finally, we would like to notice that the construction here presented is essentially *topological*, in the sense that its main ingredient is simply index theory. All the geometric structures used in Section 2 (spin structure, positivity of scalar curvature, hyperkähler metric, etc.) were needed either because a particular differential operator was used (i.e. the Dirac operator), or because we selected those objects (i.e. anti-self-dual connection over hyperkähler manifolds) that yielded very particular transforms (anti-self-dual connections).

One can conceive, for instance, a similar construction either based on a different pseudodifferential elliptic operator, other than the Dirac operator, or allowing for classes in $K(T)$, rather than actual vector bundles over the parameter space. The author thus believes that a much more general construction in a "K-theory with connections", akin to the Fourier–Mukai transform in the derived category of coherent sheaves over algebraic varieties, underlies the construction here presented. We hope to address this issue in a future paper.

3. Examples

As we mentioned in Section 1, several examples of the Nahm transform have been described in the literature, and we now take some time to revise them.

3.1. *Invariant instantons on* $\mathbb{R}^4$ *and dimensional reduction*

First, we consider the case of translation invariant instantons on $\mathbb{R}^4$, for which the Nahm transform was first developed. Let Λ be a subgroup of translations $\mathbb{R}^4$; the dual group

$$\Lambda^* = \{\alpha \in (\mathbb{R}^4)^* | \alpha(\lambda) \in \mathbb{Z}\, \forall \lambda \in \Lambda\}$$

can be regarded as a subgroup of translations $(\mathbb{R}^4)^*$. With this in mind, we set $M = \mathbb{R}^4/\Lambda$, and $T = (\mathbb{R}^4)^*/\Lambda^*$.

A point $\xi \in T$ can be canonically identified with the flat connection $\mathrm{i} \cdot \xi$, with ξ being regarded as a (constant) one-form on M, on a topologically trivial line bundle over M. Thus all of the Nahm transforms included in this example are *flat*. Conversely, it is easy to see that a point $x \in M$ can also be thought as the flat connection $\mathrm{i} \cdot x$ on a topologically trivial line bundle over T.

At this point it might be useful to briefly remind the reader of the various gauge theoretical equations obtained from the anti-self-duality equations via dimensional reduction. A connection on a Hermitian vector bundle over $\mathbb{R}^4$ of rank n can be regarded as one-form

$$A=\sum_{k=1}^{4}A_k(x_1,\dots,x_4)\,\mathrm{d}x^k,\qquad A_k:\mathbb{R}^4\to\mathfrak{u}(n).$$

Assuming that the connection components A_k are invariant under translation in one direction, say x_4, we can think of $\underline{A}=\sum_{k=1}^{3}A_k(x_1,x_2,x_3)\,\mathrm{d}x^k$ as a connection on a Hermitian vector bundle over $\mathbb{R}^3$, with the fourth component $\phi=A_4$ being regarded as a bundle endomorphism (the Higgs field). In this way, the anti-self-duality equations (1) reduce to the so-called Bogomolny (or monopole) equation:

$$F_{\underline{A}}=*\,\mathrm{d}\phi, \tag{10}$$

where $*$ is the Euclidean Hodge star in dimension 3.

Now assume that the connection components A_k are invariant under translation in two directions, say x_3 and x_4. Consider $\underline{A}=\sum_{k=1}^{2}A_k(x_1,x_2)\,\mathrm{d}x^k$ as a connection on a Hermitian vector bundle over $\mathbb{R}^2$, with the third and fourth components combined in a complex bundle endomorphism: $\Phi=(A_3+\mathrm{i}\cdot A_4)(\mathrm{d}x_1-\mathrm{i}\cdot\mathrm{d}x_2)$. The anti-self-duality equations (1) are then reduced to the so-called Hitchin's equations:

$$F_{\underline{A}}=-[\Phi,\Phi^*],\qquad \bar{\partial}_{\underline{A}}\Phi=0. \tag{11}$$

Finally, assume that the connection components A_k are invariant under translation in three directions, say x_2, x_3 and x_4. After gauging away the first component A_1, the anti-self-duality equations (1) reduce to the so-called Nahm's equations:

$$\frac{\mathrm{d}T_k}{\mathrm{d}x_1}+\frac{1}{2}\sum_{j,l}\epsilon_{kjl}[T_j,T_l]=0,\qquad j,k,l=\{2,3,4\}. \tag{12}$$

Roughly speaking, *the Nahm transform yield a* 1–1 *correspondence between Λ-invariant instantons on* $\mathbb{R}^4$ *and Λ^*-invariant instantons on* $(\mathbb{R}^4)^*$. Except for the case $\Lambda=\mathbb{Z}^4$, both M and T are non-compact. This case is also the only one that relates instantons to instantons, and does not involve a dimensional reduction on either side of the correspondence.

There are plenty of examples of the Nahm transform for translation invariant instantons available in the literature, namely:

1. The trivial case $\Lambda=\{0\}$ is closely related to the celebrated ADHM construction of instantons, as described by Donaldson and Kronheimer [13]; in this case, $\Lambda^*=(\mathbb{R}^4)^*$ and an instanton on $\mathbb{R}^4$ corresponds to some algebraic datum (ADHM datum).
2. $\Lambda=\mathbb{R}$ gives rise to monopoles, extensively studied by Hitchin [18], Donaldson [12], Hurtubise and Murray [19] and Nakajima [26], among several others; here, $\Lambda^*=\mathbb{R}^3$, and the transformed object is, for SU(2) monopoles, an analytic solution of Nahm's equations defined over the open interval $(-1,1)$ and with simple poles at the end-points.
3. If $\Lambda=\mathbb{Z}^4$, this is the Nahm transform of Schenk [30], Braam and van Baal [8] and Donaldson and Kronheimer [13], defining a correspondence between instantons over two dual four-dimensional tori.
4. $\Lambda=\mathbb{Z}$ correspond to the so-called calorons, studied by Nahm [25], van Baal [31] and others (see [28] and the references therein); the transformed object is the solution of Nahm-type equations on a circle.

5. The case $\Lambda = \mathbb{Z}^2$ (doubly periodic instantons) has been analysed in great detail by the author [20–22] and Biquard [7]. here, $\Lambda^* = \mathbb{Z}^2 \times \mathbb{R}^2$, and the Nahm transform gives a correspondence between doubly periodic instantons and certain *tame* solutions of Hitchin's equations on a two-torus.
6. $\Lambda = \mathbb{R} \times \mathbb{Z}$ gives rise to the periodic monopoles considered by Cherkis and Kapustin [9]; in this case, $\Lambda^* = \mathbb{Z} \times \mathbb{R}$, and the Nahm dual data is given by certain solutions of Hitchin's equations on a cylinder.

In the following two sections we will take a closer look at periodic instantons and monopoles.

3.2. Periodic instantons

Let us now focus on the case of periodic instantons, that is $\Lambda = \mathbb{Z}^d$ and $M = \mathbb{T}^d \times \mathbb{R}^{4-d}$, where $d = 1, 2, 3, 4$; in these cases, $\Lambda^* = \mathbb{Z}^d \times \mathbb{R}^{4-d}$ and $T = \hat{\mathbb{T}}^d$. Other useful accounts of the Nahm transform for periodic instantons in the physical literature can be found at [14,16], for example.

In all the above examples, the general statement one can prove is that there exists a 1–1 correspondence between instantons over M and singular solutions of the dimensionally reduced anti-self-duality equations over T.

Indeed, the correspondence is established just as explained in the previous section, with some minor modifications needed to deal with the non-compactness of M. Let $T_F(E, A)$ denote set of all points $\xi \in T = \mathbb{T}^d$ (regarded as a trivial bundle with flat connection) such that the Dirac operator coupled with the tensor connection $A_\xi = A \otimes \mathbf{1} + \mathbf{1} \otimes \xi$ is Fredholm. Roughly speaking, $T_F(E, A)$ depends only on the asymptotic behaviour of the connection A, and not on the topological invariants of the bundle E; it consists of T minus finitely many points.

With this in mind, $T_F(E, A)$ can be regarded as parametrising a family of elliptic Fredholm operators D_{A_ξ} on the bundle $E \to M$. Given that M is flat as a Riemannian manifold, the Weitzenböck formula (2) can be used to show that $\ker D_{A_\xi} = 0$ for all $\xi \in T_F(E, A)$, so that $\hat{E} = -\mathrm{Index}\{D_{A_t}\}$ is a Hermitian vector bundle over $T_F(E, A)$. Now $\hat{E}$ can be lifted to a bundle over (a open subset of) $(\mathbb{R}^4)^*$. A connection $\hat{A}$ on the lifted bundle is defined via the projection formula (4), and $\hat{A}$ can be seen to be anti-self-dual via the hyperkähler rotation argument in Proposition 5. Now $\hat{A}$ descends to the quotient $T_F(E, A)$, and thus defines a solution of the dimensionally reduced anti-self-duality equations. Finally, this procedure is invertible, since M can also be regarded as parametrising trivial line bundles with flat connections over T.

This simplified statement is still not proven in full generality; only the compact cases $d = 4$ and $d = 2$ have been fully described in the literature. The compact case ($d = 4$) is the easiest one, and it is closely related to the celebrated Fourier–Mukai transform in algebraic geometry; see for instance [8,13]. A precise result in this case is as follows.

Theorem 7. *There exists a* 1–1 *correspondence between the following objects*:

- $\mathrm{SU}(n)$ *instantons over* $M = \mathbb{T}^4$, *of charge* k;
- $\mathrm{SU}(k)$ *instantons over* $M = \hat{\mathbb{T}}^4$, *of charge* n.

The analysis of the non-compact cases ($d = 1, 2, 3$) involve, as we mentioned above, a careful study of the instanton's asymptotic behaviour, checking that the coupled Dirac operator is indeed Fredholm and correctly applying the Fredholm theory. The key issue to understand is how the asymptotic data gets transformed.

Doubly periodic instantons have been extensively studied by the author in [7,20–22]. Here is the full statement of the correspondence, taking into account the asymptotic behaviour of instantons and the singularities of the transformed Nahm data, in the simplest case of SU(2) gauge group.

Theorem 8. *There exists a* 1–1 *correspondence between the following objects*:

- *An anti-self-dual* SU(2) *connection A on a rank* 2 *vector bundle $E \to \mathbb{T}^2 \times \mathbb{R}^2$ such that*

$$\frac{1}{8\pi^2}\int_{\mathbb{T}^2\times\mathbb{R}^2} |F_A|^2 = k,$$

  and whose asymptotic expansion, up to gauge transformations, as $r \to \infty$ and for some $\xi = \lambda_1 + \mathrm{i}\lambda_2 \in \hat{\mathbb{T}}^2$, $\mu = \mu_1 + \mathrm{i}\mu_2 \in \mathbb{C}$, *and* $\alpha \in [0, 1/2)$, *is given either by*

$$\mathrm{i}\begin{pmatrix} a_0 & 0 \\ 0 & -a_0 \end{pmatrix} + \mathrm{O}(r^{-1-\delta})$$

$$\textit{with}\;\; a_0=\lambda_1\,\mathrm{d}x+\lambda_2\,\mathrm{d}y+(\mu_1\cos\theta-\mu_2\sin\theta)\frac{\mathrm{d}x}{r}+(\mu_1\sin\theta+\mu_2\cos\theta)\frac{\mathrm{d}y}{r}+\alpha\,\mathrm{d}\theta,$$

  if $\xi, \mu, \alpha \neq 0$; *or, if* $\xi, \mu, \alpha = 0$, *by*

$$\mathrm{i}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\mathrm{d}\theta}{\ln r^2} + \frac{1}{r\ln r^2}\begin{pmatrix} 0 & -\bar{a}_0 \\ a_0 & 0 \end{pmatrix} + \mathrm{O}(r^{-1}(\ln r)^{-1-\delta})$$

$$\textit{with}\;\; a_0 = -\mathrm{e}^{\mathrm{i}\theta}(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y).$$

- *An Hermitian connection B on a rank k Hermitian vector bundle $V \to \hat{\mathbb{T}}^2 \setminus \{\pm\xi\}$ and a skew-Hermitian bundle endomorphism Φ* (*the Higgs field*) *satisfying Hitchin's equations, and having at most simple poles at $\pm\xi$. Moreover, the residue of Φ either has rank one, if $\xi \neq -\xi$, or has rank two, if $\xi = -\xi$, with $\pm\mu$ being the only nonzero eigenvalues*; *similarly the monodromy of the connection B near the punctures is semisimple, with either only one nontrivial eigenvalue* $\exp(\mp 2\pi\mathrm{i}\alpha)$, *or two if $\xi_0 = -\xi_0$*.

The main feature of the above statement is the matching of the instanton's asymptotic behaviour with the Nahm transformed data's singularity behaviour.

It is certainly possible to generalise this correspondence for higher rank (see [21]), but that would require a much more lengthy analysis of both the asymptotic behaviour of A and the singularity data of (B, Φ). It suffices to say that the while the instanton number k determines the rank of the Nahm transformed bundle V, the rank of the original instanton A determines the number of poles of the transformed Higgs field Φ (counted according with the rank of its residues).

One expects similar statements to hold also in the cases $d = 1$ (calorons) and $d = 3$ (spatially periodic instantons); although the general features of the Nahm transform in these cases are certainly known [23,26,31], a complete statement showing how the instantons asymptotic behaviour gets translated into the singularity behaviour for the Nahm transformed data is still missing.

Some positive results are available for calorons. An L^2-index theorem for the Dirac operator coupled to calorons has been established by Nye and Singer [29], while the Nahm transform itself has been studied by Nye in his thesis [28]. Nye has identified the appropriate asymptotic behaviour for calorons, and the corresponding singularity behaviour for the Nahm data on the dual circle S^1. He has also constructed the Nahm transform from calorons to Nahm data on S^1 and from Nahm data on S^1 to calorons; however, he has not proved that these are mutually inverse, something that can probably be done using holomorphic geometry and the cohomological argument of [9,13,21].

Moreover, it is also reasonable to expect that the above results for $d = 2, 4$ (as well as the expected ones for $d = 1, 3$) can be adapted to deal with $\mathbb{Z}_p$-*equivariant* instantons on $\mathbb{T}^d \times \mathbb{R}^{n-d}$.

3.3. *Periodic monopoles*

The case of periodic monopoles, that is $\Lambda = \mathbb{Z}^d \times \mathbb{R}$, where $d = 0, 1, 2$. As in the case of instantons, the Nahm transform yields a correspondence between the following objects:

- monopoles on $M = \mathbb{T}^d \times \mathbb{R}^{3-d}$;
- solutions of the dimensionally reduced anti-self-duality equations over $T = \mathbb{T}^d \times \mathbb{R}$.

The non-periodic case ($d = 0$) was first described by Hitchin in his classical paper [18] in the simplest case of gauge group SU(2), and later generalised by Hurtubise and Murray [19] to include all classical groups.

Theorem 9. *There exists a* 1–1 *correspondence between the following objects*:

- *An* SU(2) *connection A on a rank* 2 *vector bundle* $E \to \mathbb{R}^3$ *and a skew-Hermitian bundle endomorphism* Φ (*the Higgs field*) *satisfying the Bogomolny equation* (10), *and whose asymptotic expansion as* $r \to \infty$ *is given by*, *up to gauge transformations and for some positive integer k* (*the monopole number*):
$$\Phi \sim \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \cdot \left(1 - \frac{k}{2r}\right) + \mathrm{O}(r^{-2}), \quad |\nabla_A \Phi| \sim \mathrm{O}(r^{-2}) \quad \textit{and} \quad \frac{\partial |\Phi|}{\partial r} \sim \mathrm{O}(r^{-2}).$$
- *An Hermitian connection* ∇ *on a rank k Hermitian vector bundle V over the open interval* $I = (-1, 1)$ *and three skew-Hermitian bundle endomorphisms* T_a ($a = 1, 2, 3$) *satisfying Nahm's equations* (12), *and having at most simple poles at* $t = \pm 1$, *but are otherwise analytic. Moreover, the residues of* (T_1, T_2, T_3) *define an irreducible representation of* $\mathfrak{su}(2)$ *at each pole*.

The case of periodic monopoles ($d = 1$) is studied by in detail Cherkis and Kapustin [9].

Theorem 10. *There exists a* 1–1 *correspondence between the following objects*:

- *An* SU(2) *connection* A *on a rank* 2 *vector bundle* $E \to S^1 \times \mathbb{R}^2$ *and a skew-Hermitian bundle endomorphism* ϕ (*the Higgs field*) *satisfying the Bogomolny equation* (10), *and whose asymptotic expansion as* $r = |x| \to \infty$ *is given by, up to gauge transformations and for some positive integer* k (*the monopole number*) *and parameters* $v, w \in \mathbb{R}$:

$$A \sim w + \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \cdot \frac{k}{2\pi}\theta + \mathrm{O}(r^{-1}), \qquad \phi \sim v + \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \cdot \frac{k}{2\pi}\log r + \mathrm{O}(1),$$
$$|\nabla_A \Phi| \sim \mathrm{O}(r^{-1}) \quad \textit{and} \quad \frac{\partial|\Phi|}{\partial r} \sim \mathrm{O}(r^{-2}).$$

- *An Hermitian connection* B *on a rank* k *Hermitian vector bundle* $V \to \hat{S}^1 \times \mathbb{R} \simeq \mathbb{C}^*$ *and* Φ *satisfying Hitchin's equations* (11), *and whose asymptotic expansion as* $s \to \infty$ *are given by, up to gauge transformations*:

$$|F_B| \sim \mathrm{O}(|s|^{-3/2}), \qquad \mathrm{Tr}(\Phi(s)^{\alpha}) \text{ is bounded for } \alpha = 1, 2, \ldots, k-1$$
$$\textit{and} \quad \det \Phi(s) \sim \mathrm{e}^{-2\pi(v+\mathrm{i}w)} \cdot \mathrm{O}(\mathrm{e}^{\pm 2\pi s}).$$

A careful study of doubly periodic monopoles (the $d = 2$ case) is still lacking. It is interesting to note that the Nahm transform of doubly periodic monopoles is *self-dual*, in the sense that $M = T = \mathbb{T}^2 \times \mathbb{R}$; in other words, the Nahm transform takes doubly periodic monopoles into (singular) doubly periodic monopoles, probably permutating rank and charge.

3.4. K3 surfaces

A very interesting example of a *non-flat* Nahm transform was described by Bartocci et al. [4,5]. Let M be a reflexive $K3$ surface, which is defined by the following requirements:

1. M admits a Kähler form ω whose cohomology class H satisfies $H^2 = 2$;
2. M admits a holomorphic line bundle L whose Chern class $\ell = c_1(L)$ is such that $\ell \cdot H = 0$ and $\ell^2 = -12$;
3. if D is the divisor of a nodal curve on M, one has $D \cdot H > 2$.

Now let T be the moduli space of instantons of rank 2 with determinant line bundle L (so that $c_1 = \ell$) and $c_2 = -1$ over M; it can be shown that T is isomorphic to M as a complex algebraic variety [4]. Since both M and T are hyperkähler manifolds, Nahm transform takes instantons over M into instantons over T. Furthermore, under appropriate circumstances, the transform is invertible, and one obtains in particular the following result [4,5].

Theorem 11. *There exists a* 1–1 *correspondence between the following objects* ($n \geq 2$ *and* $k \geq 1$):

- SU(n) *instantons of charge* k *over* M;
- $U(2n+k)$ *instantons of charge* k *over* T, *with first Chern class given by* $k\hat{\ell}$.

Finally, we would like to point out that a similar result also holds for hyperkähler ALE four-manifolds; a preliminary version was announced in [6] (see also [17]).

3.5. First new example: doubly periodic instantons

Let us now proceed to describe two new examples of *non-flat* Nahm transforms. The second one, described below, is particularly interesting, for it is the only example in which M is not a hyperkähler four-manifold.

Our first new example of a non-flat Nahm transform is based on the observation that, once asymptotic parameters (ξ, μ, α) are fixed, the moduli space $\mathcal{M}_{(1,\xi,\mu,\alpha)}$ of charge one SU(2) doubly periodic instantons (as described in Theorem 8) is just $\mathbb{T}^2 \times \mathbb{R}^2$ with the flat metric [7].

Thus set $M = \mathbb{T}^2 \times \mathbb{R}^2$ and $T = \mathcal{M}_{(1,\xi,\mu,\alpha)} = \mathbb{T}^2 \times \mathbb{R}^2$; let $E \to M$ be a Hermitian vector bundle of rank n, and let A be an anti-self-dual connection on E. Denote the points of T by the pair (F, B) consisting of a rank 2 Hermitian vector bundle F and an anti-self-dual connection B. If the asymptotic state of the connection A does not contain ξ, then the twisted connection $A_B = A \otimes \mathbf{1} + \mathbf{1} \otimes B$ contains no flat factors at infinity, and the Dirac operators $D^{\pm}_{A_B}$ are Fredholm [21]. This means that the Nahm transformed bundle with connection $(\hat{E}, \hat{A}) \to T$ are well defined, according to procedure in Section 2. Using the hyperkähler rotation method of Proposition 5, one sees that $\hat{A}$ is also anti-self-dual.

Clearly, M can also be regarded as a moduli space of instantons on T, so there is a Nahm transform that transforms instantons on T into instantons M. It seems reasonable to conjecture that these transforms are the inverse of one another.

3.6. Second new example: instantons over the four-sphere

Let us now briefly analyse the Nahm transform for the simplest possible compact spin four-manifold with non-negative scalar curvature. Let $M = S^4$ be the round four-dimensional sphere, and let T be the moduli space of SU(2) instantons over S^4 with charge one; as a Riemannian manifold, T is a hyperbolic five-ball $\mathbb{B}^5$ [15].

So let $E \to S^4$ be a complex vector bundle of rank $n \geq 2$, provided with an instanton A of charge $k \geq 1$. Nahm transform gives a bundle $\hat{E} \to B^5$ of rank $2k + r$, by the index formula (6). Since $\mathbb{B}^5$ is simply connected, this is the only nontrivial topological invariant of the transformed bundle. This illustrates the wide range of possibilities for a Nahm transform beyond the confines of hyperkähler geometry.

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